8. Integracija nekih iracionalnih f-ja

Ovu lekciju možemo podjeliti na pet vrsta integrala

I. $\int R(x, x^{\alpha}, x^{\beta}, \dots)\,dx$ gdje je R racionalna f-ja,

$\alpha = \frac{m_1}{n_1}$, $\beta = \frac{m_2}{n_2}$ - uvodimo smjenu $x = t^k$ gdje je k broj takav da u novodobijenom integralu na promjenjivoj t ostanu samo cijeli stepeni.

$$\int R(x, (ax+b)^{\alpha}, (ax+b)^{\beta})\,dx \quad \text{ili}$$

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\alpha}, \left(\frac{ax+b}{cx+d}\right)^{\beta}, \dots\right)$$

rješavamo uvođenjem smjene $ax+b = t^k$ ili

$$\frac{ax+b}{cx+d} = t^k.$$

II. $\int R(x, \sqrt{a^2-x^2})\,dx$ - uvodimo smjenu $x = a\sin t$;

$\int R(x, \sqrt{a^2+x^2})\,dx$ - uvodimo smjenu $x = a\,tg\,t$;

$\int R(x, \sqrt{x^2-a^2})\,dx$ - uvodimo smjenu $x = \frac{a}{\cos t}$

III. $\int x^m (a+bx^n)^p\,dx$ (integral binomnog diferencijala)

a) kada je $p \in \mathbb{Z}$ (p cijeli broj) - uvodimo smjenu $x = t^s$ gdje je $s = NZS(m, n)$ (najmanji zajednički sadržalac) ili (ako je $p \in \mathbb{Z}$) razlažemo na dijelove pomoću binomne formule

b) kada je $\frac{m+1}{n} \in \mathbb{Z}$ - uvodimo smjenu $a+bx^n = t^r$ gdje je r nazivnik od p

c) kada je $\frac{m+1}{n} + p \in \mathbb{Z}$ - uvodimo smjenu $a+bx^n = x^n t^r$ gdje je r nazivnik broja p

IV. $\int \frac{P_n(x)}{\sqrt{V}} dx$ gdje je $P_n(x)$ polinom n-tog stepena, a $V = ax^2 + bx + c$. Ovaj integral možemo odrediti po formuli

$$\int \frac{P_n(x)}{\sqrt{V}} dx = (A_1 x^{n-1} + A_2 x^{n-2} + \dots + A_n)\sqrt{V} + B \int \frac{dx}{\sqrt{V}}$$

gdje su $A_1, A_2, \dots, A_n, B$ brojevi koje dobijemo iz sistema jednačina, a sistem jednačina dobijemo tako što datu formulu prvo diferenciramo a onda dobijeni diferencijal pomnožimo sa $\sqrt{V}$. (Metoda Ostrogradski)

V $\int \frac{(Ax+B)\,dx}{(x-\alpha)\sqrt{ax^2+bx+c}}$ - uvodimo smjenu $x - \alpha = \frac{1}{t}$

Odrediti integrale

a) $\displaystyle\int \frac{1+\sqrt[4]{x}}{x+\sqrt{x}}\,dx$ b) $\displaystyle\int \frac{1}{x^2}\sqrt{\frac{1+x}{x}}\,dx$ c) $\displaystyle\int \frac{\sqrt{(4-x^2)^3}}{x^6}\,dx$

d) $\displaystyle\int \frac{dx}{x^2\sqrt[3]{(1+x^3)^5}}$ e) $\displaystyle\int \frac{2x^2-x-5}{\sqrt{x^2-2x}}\,dx$ f) $\displaystyle\int \frac{dx}{(x-1)\sqrt{1-x^2}}$

Rj.

a) $$\int \frac{1+\sqrt[4]{x}}{x+\sqrt{x}}\,dx = \left|\begin{array}{l} x=t^4 \\ dx=4t^3dt \end{array}\right| = \int \frac{1+t}{t^4+t^2}\,4t^3\,dt$$

$$= 4\int \frac{t^2+t\overset{+1-1}{}}{t^2+1}\,dt = 4\int \left(1+\frac{t-1}{t^2+1}\right)dt =$$

$$= 4\left(\int dt + \int \frac{\overbrace{\tfrac{1}{2}d(t^2+1)}^{=t\,dt}}{t^2+1} - \int \frac{dt}{t^2+1}\right) =$$

$$= 4t + 2\ln(t^2+1) - 4\,\mathrm{arc\,tg}\,t + C$$

$$= 4\sqrt[4]{x} + 2\ln(1+\sqrt{x}) - 4\,\mathrm{arc\,tg}\sqrt[4]{x} + C$$

b) $\displaystyle\int \frac{1}{x^2}\sqrt{\frac{1+x}{x}}\,dx$ Prema pravilu I: $\displaystyle\frac{1+x}{x}=t^2$

Odavde imamo $\displaystyle\frac{1\cdot x-(1+x)\cdot 1}{x^2}\,dx = 2t\,dt \Rightarrow \frac{-1}{x^2}\,dx = 2t\,dt$

$$\Rightarrow \frac{dx}{x^2} = -2t\,dt$$

$$\boxed{\frac{1+x}{x}=t^2 \Rightarrow 1+x=t^2\cdot x \Rightarrow (1-t^2)x=-1 \Rightarrow x=\frac{1}{t^2-1}}$$

$$\int \frac{1}{x^2}\sqrt{\frac{1+x}{x}}\,dx = \left|\begin{array}{l} \frac{1+x}{x} = t^2 \\ \frac{dx}{x^2} = -2t\,dt \end{array}\right| = \int (-2)\,t\cdot t\,dt =$$

$$= -2\int t^2\,dt = -\frac{2}{3}t^3 + C = -\frac{2}{3}\sqrt{\left(\frac{1+x}{x}\right)^3} + C$$

c) $$\int \frac{\sqrt{(4-x^2)^3}}{x^6}\,dx = \left|\begin{array}{l} x = 2\sin t \\ dx = 2\cos t\,dt \end{array}\right| = \int \frac{\overbrace{\sqrt{(4-4\sin^2 t)^3}}^{\sqrt{4^3(1-\sin^2 t)^3}}}{64\sin^6 t}\,2\cos t\,dt$$

$$= \frac{2^3\cdot 2}{64}\int \frac{\cos^3 t}{\sin^6 t}\cos t\,dt = \frac{1}{4}\int \frac{\cos^4 t}{\sin^6 t}\,dt = \frac{1}{4}\int ctg^4 t\cdot\frac{dt}{\sin^2 t}$$

$$= \left|\begin{array}{l} d(ctg\,t) = \frac{-dt}{\sin^2 t} \\ \frac{dt}{\sin^2 t} = -d(ctg\,t) \end{array}\right| = -\frac{1}{4}\int ctg^4 t\,d(ctg\,t) =$$

$$= -\frac{1}{4}\cdot\frac{1}{5}ctg^5 t + C = \left|\begin{array}{l} ctg^5 t = \frac{\cos^5 t}{\sin^5 t} = \frac{(\cos^2 t)^{\frac{5}{2}}}{(\sin^2 t)^{\frac{5}{2}}} = \\ = \frac{(1-\sin^2 t)^{\frac{5}{2}}}{(\sin^2 t)^{\frac{5}{2}}} = \frac{\left(1-\frac{x^2}{4}\right)^{\frac{5}{2}}}{\left(\frac{x^2}{4}\right)^{\frac{5}{2}}} = \\ = \frac{(4-x^2)^{\frac{5}{2}}}{(x^2)^{\frac{5}{2}}} = \frac{\sqrt{(4-x^2)^5}}{x^5} \end{array}\right| = C - \frac{\sqrt{(4-x^2)^5}}{20x^5}$$

d) $$\int \frac{dx}{x^2\sqrt[3]{(1+x^3)^5}}$$ ovo je integral binomnog diferencijala

$m=-2,\ n=3,\ p=-\frac{5}{3}$

$$\frac{m+1}{n} = \frac{-2+1}{3} = -\frac{1}{3} \notin \mathbb{Z} \qquad \frac{m+1}{n} + p = -\frac{1}{3} - \frac{5}{3} = -\frac{6}{3} = -2 \in \mathbb{Z}$$

Prema pravilu III uvodimo smjenu $1+x^3 = x^3 z^3$

$$1+x^3 = x^3 z^3$$

$$x^3 z^3 - x^3 = 1$$

$$(z^3-1)\,x^3 = 1$$

$$x^3 = \frac{1}{z^3-1}$$

$$x = \frac{1}{(z^3-1)^{\frac{1}{3}}}$$

$$1+x^3 = \frac{z^3}{z^3-1}$$

$$x^2\sqrt[3]{(1+x^3)^5} = \frac{1}{(z^3-1)^{\frac{2}{3}}}\sqrt[3]{\left(\frac{z^3}{z^3-1}\right)^5} =$$

$$= \frac{1}{(z^3-1)^{\frac{2}{3}}}\cdot\frac{z^5}{(z^3-1)^{\frac{5}{3}}} = \frac{z^5}{(z^3-1)^{\frac{7}{3}}}$$

$$dx = \left((z^3-1)^{-\frac{1}{3}}\right)' dz = -\frac{1}{3}\cdot(z^3-1)^{-\frac{4}{3}}\cdot 3z^2\, dz =$$

$$= \frac{-z^2}{(z^3-1)^{\frac{4}{3}}}\, dz$$

$$\int \frac{dx}{x^2\sqrt[3]{(1+x^3)^5}} = \left| \begin{array}{c} 1+x^3 = x^3 z^3 \\ \vdots \end{array} \right| = \int \frac{-z^2}{(z^3-1)^{\frac{4}{3}}}\cdot\frac{(z^3-1)^{\frac{7}{3}}}{z^5}\, dz =$$

$$= -\int \frac{z^3-1}{z^3}\, dz = \int \frac{1-z^3}{z^3}\, dz = \int z^{-3} dz - \int dz = \frac{z^{-2}}{-2} - z + C$$

$$= -\frac{1}{2z^2} - z + C = \left| \begin{array}{l} z^3 = x^{-3}+1 \\ z = \sqrt[3]{1+\frac{1}{x^3}} \end{array} \right| = -\frac{1}{2\sqrt[3]{\left(1+\frac{1}{x^3}\right)^2}} - \sqrt[3]{1+\frac{1}{x^3}} + C$$

e) $\displaystyle\int \frac{2x^2-x-5}{\sqrt{x^2-2x}}\, dx$ Prema pravilu IV (što je još poznato pod imenom metoda Ostrogradskog) imamo

$$I = \int \frac{2x^2-x-5}{\sqrt{x^2-2x}}\, dx = (Ax+B)\sqrt{x^2-2x} + D\int \frac{dx}{\sqrt{x^2-2x}} \quad \Big/ \frac{d}{dx}$$

$$\frac{2x^2-x-5}{\sqrt{x^2-2x}} = A\sqrt{x^2-2x} + (Ax+B)\frac{2x-2}{2\sqrt{x^2-2x}} + \frac{D}{\sqrt{x^2-2x}} \quad \Big/ \cdot\sqrt{x^2-2x}$$

$$2x^2-x-5 = A(x^2-2x)+(Ax+B)(x-1)+D$$

$$2x^2-x-5 = 2Ax^2+(B-3A)x+(D-B)$$

Izjednačavamo koeficijente uz isti stepen

x^2: $\quad 2A=2 \quad\Rightarrow\quad A=1$

x: $\quad B-3A=-1 \qquad B=2$

x^0: $\quad \underline{D-B=-5} \qquad D=-3$

$$I=(x+2)\sqrt{x^2-2x}-3\int\frac{dx}{\sqrt{x^2-2x}}$$

$$x^2-2x=x^2-2\cdot x\cdot 1+1-1 = (x-1)^2-1$$

$$\int\frac{dx}{\sqrt{x^2-2x}}=\int\frac{d(x-1)}{\sqrt{(x-1)^2-1}}=\ln\left|x-1-\sqrt{(x-1)^2-1}\right|$$

Prema tome

$$\int\frac{2x^2-x-5}{\sqrt{x^2-2x}}\,dx=(x+2)\sqrt{x^2-2x}-3\ln\left|x-1-\sqrt{(x-1)^2-1}\right|+C$$

f) $\displaystyle\int\frac{dx}{(x-1)\sqrt{1-x^2}}$ Prema pravilu $\underline{V}$ uvodimo smjenu $x-1=\frac{1}{t}$

$dx=-\frac{1}{t^2}\,dt$

$$I=\int\frac{dx}{(x-1)\sqrt{1-x^2}}=\left|\begin{array}{l}x-1=\frac{1}{t}\\ dx=-\frac{1}{t^2}dt\end{array}\right|=\int\frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{1-\underbrace{\left(\frac{1}{t}+1\right)^2}_{\frac{1}{t^2}+\frac{2}{t}+1}}}=$$

$-1-2t>0$

$2t<-1$

$t<-\frac{1}{2}$

$|t|=-t$

$$=\int\frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{-\frac{1+2t}{t^2}}}=-\int\frac{\frac{dt}{t}}{\frac{\sqrt{-1-2t}}{|t|}}=-\int\frac{|t|\,dt}{t\sqrt{-1-2t}}$$

zato što je $\sqrt{t^2}=|t|$

$$=\int\frac{dt}{\sqrt{-1-2t}}=\int(-1-2t)^{-\frac{1}{2}}\left(-\frac{1}{2}\right)d(-1-2t)=-\frac{1}{2}\cdot\frac{(-1-2t)^{\frac{1}{2}}}{\frac{1}{2}}+C=C-\sqrt{-1-\frac{2}{x-1}}$$

Zadaci za vježbu

1. $\displaystyle\int \frac{dx}{(1+\sqrt[3]{x})\sqrt{x}}$ 2. $\displaystyle\int x\sqrt{3-x}\,dx$ 3. $\displaystyle\int \frac{1}{x}\sqrt{\frac{x-2}{x}}\,dx$

4. $\displaystyle\int \frac{dx}{\sqrt{(5-x^2)^3}}$ 5. $\displaystyle\int \frac{\sqrt{1+x^2}}{x^2}\,dx$ 6. $\displaystyle\int x^2\sqrt{4-x^2}\,dx$

7.* $\displaystyle\int \frac{dx}{x\sqrt{x^2-9}}$ 8.* $\displaystyle\int \frac{\sqrt[3]{(1+2x^3)^2}}{x^6}\,dx$

9. $\displaystyle\int \frac{dt}{t\sqrt{1-t^3}}$ 10. $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2+2x+3}}$

11. $\displaystyle\int \frac{x^2+4x}{\sqrt{x^2+2x+2}}\,dx$ 12.* $\displaystyle\int \frac{x^2\,dx}{\sqrt{2ax-x^2}}$

Rješenja:

1. $6\sqrt[6]{x}-6\operatorname{arctg}\sqrt[6]{x}$ 2. $0{,}4(x^3-x-6)\sqrt{3-x}$

3. $-2\sqrt{\frac{x-2}{x}}-\ln\left[|x|\left(1-\sqrt{\frac{x-2}{x}}\right)^2\right]$ 4. $\dfrac{x}{5\sqrt{5-x^2}}$ 5. $\ln(x+\sqrt{x^2+1})$
$-\dfrac{\sqrt{x^2+1}}{x}$ 6. $2\arcsin\frac{x}{2}+\frac{x}{4}(x^2-2)\sqrt{4-x^2}$ 7. $\pm\frac{1}{3}\arccos\frac{3}{x}$
gdje je + kad $x>0$, − kad $x<0$. 8. $-\frac{1}{5}x^{-5}(2x^3+1)^{\frac{5}{3}}$

9. $\dfrac{1}{3}\ln\dfrac{|1-\sqrt{1-x^3}|}{1+\sqrt{1-x^3}}$ 10. $\dfrac{x-3}{2}\sqrt{x^2+2x+3}$ 11. $\dfrac{x+5}{2}\sqrt{x^2+2x+2}-$
$-\frac{7}{2}\ln(x+1+\sqrt{x^2+2x+2})$ 12. $\dfrac{3a^2}{2}\arcsin\dfrac{x-a}{a}-\dfrac{x+3a}{2}\sqrt{2ax-x^2}$

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Integracija nekih iracionalnih funkcija)

Metoda Ostrogradskog

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx = Q_{n-1}(x)\cdot\sqrt{ax^2+bx+c} + \lambda\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

1. $I=\int \frac{3x^3}{\sqrt{x^2+4x+5}}\,dx$ Rj: $=(ax^2+bx+c)\sqrt{x^2+4x+5}+\lambda\int\frac{dx}{\sqrt{x^2+4x+5}} \quad \Big/ \frac{d}{dx}$

$$\frac{3x^3}{\sqrt{x^2+4x+5}} = (2ax+b)\sqrt{x^2+4x+5} + (ax^2+bx+c)\cdot\frac{2x+4}{2\sqrt{x^2+4x+5}} + \lambda\cdot\frac{1}{\sqrt{x^2+4x+5}} \quad \Big/ \cdot\sqrt{x^2+4x+5}$$

$$3x^3 = (2ax+b)(x^2+4x+5)+(ax^2+bx+c)(x+2)+\lambda$$

$$3x^3 = (2ax^3+8ax^2+10ax)+b(x^2+4x+5)+(ax^3+bx^2+cx)+2ax^2+2bx+2c+\lambda$$

$$3x^3 = (2a+a)x^3+(8a+b+b+2a)x^2+(10a+4b+c+2b)x+5b+2c+\lambda$$

$$3x^3 = 3ax^3+(10a+2b)x^2+(10a+6b+c)x+5b+2c+\lambda$$

uz x^3: $3a=3 \Rightarrow a=1$

uz x^2: $10a+2b=0 \Rightarrow b=-5$

uz x: $10a+6b+c=0 \Rightarrow c=20$

uz x^0: $5b+2c+\lambda=0$ $\quad \lambda=-15$

$$I=(x^2-5x+20)\sqrt{x^2+4x+5} - 15\underbrace{\int\frac{dx}{\sqrt{x^2+4x+5}}}_{I_1}$$

$x^2+4x+5 = x^2+2\cdot x\cdot 2+2^2-2^2+5=(x+2)^2+1$, $\quad I_1=\int\frac{dx}{\sqrt{(x+2)^2+1}}=\ln\left|x+2+\sqrt{(x+2)^2+1}\right|+C$

$$I=(x^2-5x+20)\sqrt{x^2+4x+5}-15\ln\left|x+2+\sqrt{(x+2)^2+1}\right|+C$$

(2.) $$I=\int\frac{3x+1}{\sqrt{2x^2-x+1}}\,dx \overset{Rj.}{=} a\sqrt{2x^2-x+1}+\lambda\int\frac{dx}{\sqrt{2x^2-x+1}} \quad \Big/ \frac{d}{dx}$$

$$\frac{3x+1}{\sqrt{2x^2-x+1}}=a\cdot\frac{4x-1}{2\sqrt{2x^2-x+1}}+\lambda\cdot\frac{1}{\sqrt{2x^2-x+1}} \quad \Big/\cdot 2\sqrt{2x^2-x+1}$$

$$6x+2=a(4x-1)+2\lambda \implies \begin{array}{l} 4a=6 \\ 2\lambda-a=2 \end{array} \qquad a=\frac{6}{4}=\frac{3}{2} \qquad \lambda=\frac{7}{4}$$

$$I=\frac{3}{2}\sqrt{2x^2-x+1}+\frac{7}{4}\int\frac{dx}{\sqrt{2x^2-x+1}}=\frac{3}{2}\sqrt{2x^2-x+1}+\frac{7}{4}I_1$$

$$2x^2-x+1=2\left(x^2-\frac{1}{2}x+\frac{1}{2}\right)=2\left(x^2-2\cdot x\cdot\frac{1}{4}+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2+\frac{1}{2}\right)=2\left[\left(x-\frac{1}{4}\right)^2+\frac{7}{16}\right]$$

$$I_1=\int\frac{dx}{\sqrt{2\left[\left(x-\frac{1}{4}\right)^2+\frac{7}{16}\right]}}=\frac{1}{\sqrt2}\int\frac{dx}{\sqrt{\left(x-\frac{1}{4}\right)^2+\frac{7}{16}}}=\left|\begin{array}{l} x-\frac{1}{4}=\frac{\sqrt7}{4}t \quad /\cdot 4 \\ dx=\frac{\sqrt7}{4}dt \\ 4x-1=\sqrt7\, t \end{array}\right|=\frac{1}{\sqrt2}\int\frac{\frac{\sqrt7}{4}dt}{\sqrt{\frac{7}{16}t^2-\frac{7}{16}}}$$

$$=\frac{1}{\sqrt2}\cdot\frac{\sqrt7}{4}\cdot\frac{4}{\sqrt7}\int\frac{dt}{\sqrt{t^2-1}}=\frac{1}{\sqrt2}\ln\left|t+\sqrt{t^2+1}\right|+C=\frac{1}{\sqrt2}\ln\left|\frac{4x-1}{\sqrt7}+\sqrt{\left(\frac{4x-1}{\sqrt7}\right)^2+1}\right|+C$$

$$I=\frac{3}{2}\sqrt{2x^2-x+1}+\frac{7}{4\sqrt2}\ln\left|\frac{4x-1}{\sqrt7}+\sqrt{\left(\frac{4x-1}{\sqrt7}\right)^2+1}\right|+C$$

Metodom Ostrogradskog rješavamo i integrale oblika

$$\int\sqrt{ax^2+bx+c}\,dx=\int\frac{ax^2+bx+c}{\sqrt{ax^2+bx+c}}\,dx \,.$$

(3.) $$\int\sqrt{x^2+1}\,dx \overset{Rj.}{=} (ax+b)\sqrt{x^2+1}+\lambda\int\frac{dx}{\sqrt{x^2+1}} \quad \Big/\frac{d}{dx}$$

$$\sqrt{x^2+1}=a\sqrt{x^2+1}+(ax+b)\frac{2x}{2\sqrt{x^2+1}}+\lambda\cdot\frac{1}{\sqrt{x^2+1}} \quad \Big/\cdot\sqrt{x^2+1}$$

$$x^2+1=a(x^2+1)+(ax^2+bx)+\lambda$$

x^2: $a+a=1$, $2a=1$, $a=\frac{1}{2}$

x: $b=0$

x^0: $a+\lambda=1$, $\lambda=1-\frac{1}{2}$, $\lambda=\frac{1}{2}$

$$\int \sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right| + C$$

4. $$\int \frac{2x^2-3x}{\sqrt{x^2-2x+5}}\,dx$$

5. $$\int \sqrt{x^2-2x-1}\,dx$$

6. $$\int \frac{x^5}{\sqrt{1-x^2}}\,dx \qquad \text{Rj: } -\frac{8+4x^2+3x^4}{15}\sqrt{1-x^2}$$

7. $$\int x^4\sqrt{1-x^2}\,dx \qquad \text{uputa: } \int \frac{x^4(1-x^2)}{\sqrt{1-x^2}}\,dx$$

$$\int R\left(x, \sqrt[n]{ax+b}\right)dx, \quad \text{smjena } ax+b = t^n$$

1. $$\int \frac{dx}{\sqrt{2x-1}-\sqrt[4]{2x-1}} = \left| \begin{array}{l} 2x-1 = t^4 \\ 2dx = 4t^3dt \quad /:2 \\ dx = 2t^3dt \\ t = \sqrt[4]{2x-1} \end{array} \right| = 2\int \frac{t^3\,dt}{\sqrt{t^4}-\sqrt[4]{t^4}} = 2\int \frac{t^3}{\underbrace{t^2-t}_{t(t-1)}}dt$$

$$= 2\int \frac{t^2}{t-1}dt = 2\int \frac{t^2-1+1}{t-1}dt = 2\int \frac{t^2-1}{t-1}dt + \int \frac{dt}{t-1} =$$

$$= 2\int \frac{(t-1)(t+1)}{t-1}dt + 2\int \frac{dt}{t-1} = 2\int (t-1)\,dt + 2\int \frac{1}{t-1}dt = 2\cdot\frac{t^2}{2} - 2t + 2\ln|t-1| + C$$

$$= \sqrt[4]{(2x-1)^2} + 2\sqrt[4]{(2x-1)} + 2\ln\left|\sqrt[4]{2x-1}-1\right| + C = \sqrt{2x-1} + 2\sqrt[4]{2x-1} + 2\ln\left|\sqrt[4]{2x-1}-1\right| + C$$

2. $$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx = \left| \begin{array}{l} x = t^4 \\ dx = 4t^3dt \\ t = \sqrt[4]{x} \end{array} \right| = \int \frac{\sqrt[3]{1+t}}{t^2}\cdot 4t^3\,dt =$$

$$= 4\int t\sqrt[3]{1+t}\,dt = \left| \begin{array}{l} 1+t = u^3 \\ dt = 3u^2du \\ u = \sqrt[3]{1+t} \\ t = u^3-1 \end{array} \right| = 4\int (u^3-1)\sqrt[3]{u^3}\cdot 3u^2\,du =$$

$$= 12\int (u^6-u^3)\,du = 12\left(\frac{u^7}{7} - \frac{u^4}{4}\right) + C = \frac{12}{7}u^7 - 3u^4 + C =$$

$$= \frac{12}{7}\sqrt[3]{(1+t)^7} - 3\sqrt[3]{(1+t)^4} + c = \frac{12}{7}\sqrt[3]{(1+\sqrt[4]{x})^7} - 3\sqrt[3]{(1+\sqrt[4]{x})^4} + C$$

3. $\int \sqrt{\frac{x+1}{x-1}}\,dx$ ako stavim smjenu $\frac{x+1}{x-1} = t^2$ dobiću

$$x+1 = t^2(x-1)$$
$$x+1 = t^2x - t^2$$
$$x - t^2x = -t^2 - 1$$
$$(1-t^2)x = -t^2-1 \quad /(-1)$$
$$x = \frac{t^2+1}{t^2-1}$$

$$dx = d\left(\frac{t^2+1}{t^2-1}\right)$$
$$dx = \frac{2t(t^2-1)-(t^2+1)\cdot 2t}{(t^2-1)^2}dt$$
$$dx = \frac{2t^3-2t-2t^3-2t}{(t^2-1)^2}dt$$
$$dx = \frac{-4t}{(t^2-1)^2}dt$$

$$I = \int t\cdot\frac{-4t}{(t^2-1)^2}dt = -4\int t\cdot\frac{t}{(t^2-1)^2}dt = -4I_1$$

$$I_1 = \int t\cdot\frac{t}{(t^2-1)^2}dt = \left| \begin{array}{ll} u = t & dv = \frac{t}{(t^2-1)^2}dt \\ du = dt & \end{array}\right.$$

$$v = \int\frac{t}{(t^2-1)^2}dt = \left|\begin{array}{l} t^2-1 = z \\ 2t\,dt = dz \\ t\,dt = \frac{dz}{2}\end{array}\right| = \int\frac{\frac{1}{2}dz}{z^2} =$$

$$= \frac{1}{2}\int z^{-2}dz = \frac{1}{2}\cdot\frac{z^{-1}}{-1} = -\frac{1}{2}\cdot\frac{1}{z} = -\frac{1}{2(t^2-1)}$$

$$= -\frac{1}{2}\cdot\frac{t}{t^2-1} + \frac{1}{2}\int\frac{dt}{t^2-1} = -\frac{1}{2}\cdot\frac{t}{t^2-1} + \frac{1}{2}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + c_1$$

$$I = -4\cdot\left(-\frac{1}{2}\cdot\frac{t}{t^2-1} + \frac{1}{4}\ln\left|\frac{t-1}{t+1}\right|\right) + c =$$

$$= 2\cdot\frac{\sqrt{\frac{x+1}{x-1}}}{\frac{x+1}{x-1}-1} - 1\ln\left|\frac{\sqrt{\frac{x+1}{x-1}}-1}{\sqrt{\frac{x+1}{x-1}}+1}\right| + C$$

4. $\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}$ Rj.: $6\sqrt[6]{x} - 3\sqrt[3]{x} + 2\sqrt{x} - 6\ln(1+\sqrt[6]{x}) + C$

5. $\int\frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}dx$ Rj.: $\ln\left|\frac{(\sqrt{x+1}-1)^2}{x+2+\sqrt{x+1}}\right| - \frac{2}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2\sqrt{x+1}+1}{\sqrt{3}} + C$

6. $\displaystyle\int \frac{dx}{(2-x)\sqrt{1-x}}$ Rj. $-2\,\mathrm{arctg}\sqrt{1-x} + C$

integrali koji se mogu riješiti racionalisanjem

1. $$I=\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\,dx$$

Rj.

$$I=\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\cdot\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}\,dx=\int\frac{x+1-2\cdot\sqrt{x+1}\cdot\sqrt{x-1}+x-1}{x+1-(x-1)}\,dx=$$

$$=\frac{1}{2}\int(2x-2\sqrt{x^2-1})\,dx=\int x\,dx-\int\sqrt{x^2-1}\,dx=\frac{x^2}{2}-I_1$$

$$I_1=\int\sqrt{x^2-1}\,dx=(ax+b)\sqrt{x^2-1}+\lambda\int\frac{dx}{\sqrt{x^2-1}} \quad \Big/ \frac{d}{dx}$$

$$\sqrt{x^2-1}=a\cdot\sqrt{x^2-1}+(ax+b)\frac{2x}{2\sqrt{x^2-1}}+\lambda\cdot\frac{1}{\sqrt{x^2-1}} \quad \Big/\sqrt{x^2-1}$$

$$x^2-1=a(x^2-1)+(ax^2+bx)+\lambda$$

x^2: $a+a=1 \Rightarrow a=\frac{1}{2}$, x: $b=0$, x^0: $-a+\lambda=-1$ $\lambda=-\frac{1}{2}$

$$I_1=\int\sqrt{x^2-1}\,dx=\frac{1}{2}\sqrt{x^2-1}-\frac{1}{2}\ln|x+\sqrt{x^2-1}|+C$$

$$I=\frac{1}{2}x^2-\frac{1}{2}\sqrt{x^2-1}+\frac{1}{2}\ln|x+\sqrt{x^2-1}|+C$$

2. $$\int\frac{x}{\sqrt{x+2}+\sqrt{x+1}}\,dx$$

Rj.

$$\int\frac{x}{\sqrt{x+2}+\sqrt{x+1}}\cdot\frac{\sqrt{x+2}-\sqrt{x+1}}{\sqrt{x+2}-\sqrt{x+1}}\,dx=\int\frac{x\sqrt{x+2}-x\sqrt{x+1}}{x+2-(x+1)}\,dx=$$

$$=\int(x\sqrt{x+2}-x\sqrt{x+1})\,dx=\int x\sqrt{x+2}\,dx-\int x\sqrt{x+1}\,dx=I_1-I_2$$

$$I_1=\int x\sqrt{x+2}\,dx=\left|\begin{array}{l} x+2=t^2\\ dx=2t\,dt\\ x=t^2-2\\ t=\sqrt{x+2}\end{array}\right|=\int(t^2-2)\,t\cdot 2t\,dt=2\int(t^4-2t^2)\,dt=2\cdot\frac{t^5}{5}-4\frac{t^3}{3}+C_1$$

$$=\frac{2}{5}\sqrt{(x+2)^5}-\frac{4}{3}\sqrt{(x+2)^3}+C_1$$

$$I_2=\int x\sqrt{x+1}\,dx=\left|\begin{array}{l} x+1=t^2\\ x=t^2-1\\ dx=2t\,dt\\ t=\sqrt{x+1}\end{array}\right|=\int(t^2-1)\cdot t\cdot 2t\,dt=2\int(t^4-t^2)\,dt=$$

$$=2\frac{t^5}{5}-2\cdot\frac{t^3}{3}+C_2=\frac{2}{5}\sqrt{(x+1)^5}-\frac{2}{3}\sqrt{(x+1)^3}+C_2$$

$$I=\frac{2}{5}\sqrt{(x+2)^5}-\frac{4}{3}\sqrt{(x+2)^3}-\frac{2}{5}\sqrt{(x+1)^5}+\frac{2}{3}\sqrt{(x+1)^3}+C$$

3.° $\displaystyle\int\frac{dx}{x-\sqrt{x^2-1}}$. Rj: $\frac{1}{2}x^2+\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln|x+\sqrt{x^2-1}|+C$.

4.° $\displaystyle\int\frac{dx}{\sqrt{x^2+1}-x}$

5.° $\displaystyle\int\frac{\sqrt{x^2+2x+2}}{x}\,dx$

$$\int\frac{Mx+N}{(x-\alpha)^n\sqrt{ax^2+bx+c}}\,dx,\quad n\in\mathbb{N},\quad M,N,a,b,c\in\mathbb{R}$$

uvodimo smjenu $x-\alpha=\frac{1}{t}$

1.° $$\int\frac{dx}{(x+1)\sqrt{x^2+x+1}}=\left|\begin{array}{l} x+1=\frac{1}{t}\\ dx=-\frac{1}{t^2}dt\\ t=\frac{1}{x+1}\end{array}\right|=\int\frac{-\frac{1}{t^2}dt}{t\sqrt{\underbrace{\left(\frac{1}{t}-1\right)^2+\frac{1}{t}-1+1}_{\frac{1}{t^2}-\frac{2}{t}+1+\frac{1}{t}}}}=-\int\frac{dt}{t\sqrt{\frac{1}{t^2}-\frac{1}{t}+1}}$$

$$=-\int\frac{dt}{\sqrt{t^2-t+1}}=\left|\begin{array}{l} t^2-t+1=\\ =t^2-2\cdot t\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1\\ =\left(t-\frac{1}{2}\right)^2+\frac{3}{4}\end{array}\right|=-\int\frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}}=\left|\begin{array}{l} t-\frac{1}{2}=\frac{\sqrt{3}}{2}z\\ dt=\frac{\sqrt{3}}{2}dz\\ \sqrt{3}z=2t-1\end{array}\right|$$

$$=-\frac{\sqrt{3}}{2}\int\frac{dz}{\sqrt{\frac{3}{4}z^2+\frac{3}{4}}}=-\frac{\sqrt{3}}{2}\cdot\frac{2}{\sqrt{3}}\int\frac{dz}{\sqrt{z^2+1}}=-\ln|z+\sqrt{z^2+1}|+C=$$

$$= -\ln\left|\frac{2t-1}{\sqrt{3}} + \sqrt{\left(\frac{2t-1}{\sqrt{3}}\right)^2+1}\right| + C = -\ln\left|\frac{\frac{2}{x+1}-1}{\sqrt{3}} + \sqrt{\frac{\left(\frac{2}{x+1}-1\right)^2}{3}+1}\right| + C$$

2.

$$I = \int \frac{dx}{(x-1)^3\sqrt{x^2+3x+1}} = \left| \begin{array}{l} x-1=\frac{1}{t} \\ dx = -\frac{1}{t^2}dt \\ x = \frac{1}{t}+1 \\ t = \frac{1}{x-1} \end{array} \right| = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t^3}\sqrt{\underbrace{\left(\frac{1}{t}+1\right)^2+3\left(\frac{1}{t}+1\right)+1}_{\frac{1}{t^2}+\frac{2}{t}+1+\frac{3}{t}+3+1}}} = -\int \frac{t\,dt}{\sqrt{\frac{1}{t^2}+\frac{5}{t}+5}}$$

$$= -\int \frac{t\,dt}{\sqrt{\frac{1+5t+5t^2}{t^2}}} = -\int \frac{t^2}{\sqrt{5t^2+5t+1}}dt = (at+b)\sqrt{5t^2+5t+1} + \lambda\int\frac{dt}{\sqrt{5t^2+5t+1}} \quad \Big/ \cdot \frac{d}{dt}$$

$$\frac{-t^2}{\sqrt{5t^2+5t+1}} = a\sqrt{5t^2+5t+1} + (at+b)\frac{10t+5}{2\sqrt{5t^2+5t+1}} + \lambda\cdot\frac{1}{\sqrt{5t^2+5t+1}} \quad \Big/ \cdot 2\sqrt{5t^2+5t+1}$$

$$-2t^2 = 2a\cdot(5t^2+5t+1) + a(10t^2+5t) + b(10t+5) + 2\lambda$$

t^2: $10a + 10a = -2$, $a = -\frac{1}{10}$

t: $10a + 5a + 10b = 0$

$15a + 10b = 0$

$10b = \frac{15}{10}$

$10b = \frac{3}{2}$

$b = \frac{3}{20}$

t^0: $2a + 5b + 2\lambda = 0$

$-\frac{2}{10} + \frac{15}{20} = -2\lambda$

$-2\lambda = \frac{11}{20}$

$\lambda = -\frac{11}{40}$

$$I = \left(-\frac{1}{10}t + \frac{3}{20}\right)\sqrt{5t^2+5t+1} - \frac{11}{40}\int\frac{dt}{\sqrt{5t^2+5t+1}}$$ SAMI ZAVRŠITI

3. $\int \frac{dx}{x^2\sqrt{x^2-x+1}}$

4. $\int \frac{(3x+2)\,dx}{(x+1)\sqrt{x^2+3x+3}}$

5. $\int \frac{dx}{x^3\sqrt{x^2+1}}$

integral binomnog diferencijala

$$\int x^m(a+bx^n)^p\,dx \quad (a,b\in\mathbb{R};\ m,n,p\in\mathbb{Q})$$

Integracija je moguća ako

1° $p\in\mathbb{Z}$, uvodimo smjenu $x=t^s$, $s=NZS(m_2,n_2)$, $m=\frac{m_1}{m_2}$, $n=\frac{n_1}{n_2}$
nazivnika od m i n

2° $\frac{m+1}{n}\in\mathbb{Z}$, uvodimo smjenu $a+bx^n=t^{p_2}$, $p=\frac{p_1}{p_2}$

3° $\frac{m+1}{n}+p\in\mathbb{Z}$, uvodimo smjenu $ax^{-n}+b=t^{p_2}$, p_2 nazivnik od p

integral binomnog diferencijala

$$\int x^m (a+bx^n)^p\,dx \quad (a,b\in\mathbb{R};\ m,n,p\in\mathbb{Q})$$

Integracija je moguća ako

$1°$ $p\in\mathbb{Z}$, uvodimo smjenu $x=t^s$, $s=NZS(m_2,n_2)$, $m=\frac{m_1}{m_2}$, $n=\frac{n_1}{n_2}$

nazivnika od m i n

$2°$ $\frac{m+1}{n}\in\mathbb{Z}$, uvodimo smjenu $a+bx^n=t^{p_2}$, $p=\frac{p_1}{p_2}$

$3°$ $\frac{m+1}{n}+p\in\mathbb{Z}$, uvodimo smjenu $ax^{-n}+b=t^{p_2}$, p_2 nazivnik od p

1_0 $$\int\frac{dx}{x^2(\sqrt{1+x^2})^3}=\int x^{-2}(1+x^2)^{-\frac{3}{2}}dx=\left|\begin{array}{l} m=-2,\ n=2,\ p=-\frac{3}{2}\\ p\notin\mathbb{Z},\ \text{nije } 1°\\ \frac{m+1}{n}=\frac{-2+1}{2}=-\frac{1}{2}\notin\mathbb{Z},\ \text{nije } 2°\\ \frac{m+1}{n}+p=-\frac{1}{2}-\frac{3}{2}=-2\in\mathbb{Z},\ 3°\end{array}\right.$$

smjena: $x^{-2}+1=t^2$

$x^{-2}=t^2-1$

$x^2=(t^2-1)^{-1}$

$x=(t^2-1)^{-\frac{1}{2}}$ $\rightarrow$ $1+x^2=1+(t^2-1)^{-1}$

$dx=-\frac{1}{2}(t^2-1)^{-\frac{3}{2}}\cdot 2t\,dt$

$dx=-t(t^2-1)^{-\frac{3}{2}}dt$

$$=\int(t^2-1)\left(1+\frac{1}{t^2-1}\right)^{-\frac{3}{2}}\cdot(-t)(t^2-1)^{-\frac{3}{2}}dt$$

$$=\int(t^2-1)\cdot\frac{t^{-3}}{(t^2-1)^{-\frac{3}{2}}}(-t)\cdot(t^2-1)^{-\frac{3}{2}}dt$$

$$=\int(-1+t^{-2})\,dt=-t-\frac{1}{t}+c=\frac{-x^{-2}-2}{\sqrt{x^{-2}+1}}+c$$

Eulerove smjene $\int R(x,\sqrt{ax^2+bx+c})\,dx$ R-racionalna f-ja

$1°$ za $a>0$ uzimamo smjenu $\sqrt{ax^2+bx+c}=\pm\sqrt{a}\,x+t$

$2°$ za $c>0$ uzimamo smjenu $\sqrt{ax^2+bx+c}=xt\pm\sqrt{c}$

$3°$ za $b^2-4ac>0$ uzimamo smjenu $\sqrt{a(x-x_1)(x-x_2)}=t(x-x_1)$

$x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}$

1_0 $$I=\int\frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}$$ Rj. $3°$

smjena $\sqrt{(x-1)(x+2)}=t(x+1)$

Integral binomnog diferencijala

$$\int x^m (a + b x^n)^p dx \quad (m, n, p \in \mathbb{Q})$$

Integracija je moguća:

1° $p \in \mathbb{Z}$

smjena: $x = t^s$

s = NZS nazivnika od m i n

2° $\frac{m+1}{n} \in \mathbb{Z}$

$a + b x^n = t^s$ – smjena

s – nazivnik od p

3° $\frac{m+1}{n} + p \in \mathbb{Z}$

$a x^{-n} + b = t^s$

s – nazivnik od p

$$\sqrt[n]{x^m} = x^{\frac{m}{n}}$$

1. $\int x^{-\frac{3}{4}} \cdot (1 + x^{\frac{1}{6}})^{-1} dx =$

$$\left| \begin{array}{l} m = -\frac{3}{4}, \; n = \frac{1}{6}, \; p = -1 \\ x = t^{12} \Rightarrow dx = 12 t^{11} dt \end{array} \right| =$$

$$= \int (t^{12})^{-\frac{3}{4}} \cdot \left(1 + (t^{12})^{\frac{1}{6}}\right)^{-1} \cdot 12 t^{11} dt =$$

$$= 12 \int t^{-9} \cdot (1 + t^2)^{-1} \cdot t^{11} dt = 12 \int \frac{t^2 + 1 - 1}{1 + t^2} dt =$$

$$= 12 \int \left(1 - \frac{1}{t^2 + 1}\right) dt = 12 \cdot \int dt - \int \frac{1}{t^2 + 1} = 12 t - \operatorname{arctg} t + C$$

$$= 12 \sqrt[12]{x} - \operatorname{arctg} \sqrt[12]{x} + C$$

2. $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}\, dx = \int x^{-\frac{2}{3}} \cdot \left(1+x^{\frac{1}{3}}\right)^{\frac{1}{2}} dx =$

$$\left|\begin{array}{ll} m = \frac{-2}{3}, \quad n = \frac{1}{3}, \quad p = \frac{1}{2} & \\ \frac{m+1}{n} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1 & 1+x^{\frac{1}{3}} = t^2 \\ 2^\circ \text{ slučaj} & \frac{1}{3} x^{\frac{1}{3}-1} dx = 2t\,dt \;/\cdot 3 \\ & x^{-\frac{2}{3}} dx = 6t\,dt \end{array}\right|$$

$= 6\int (t^2)^{\frac{1}{2}} \cdot t\, dt = 6\int t^2\, dt = \cancel{6}^{2} \cdot \frac{t^3}{\cancel{3}} + C = 2t^3 + C$

$= 2 \cdot \left(\sqrt{1+\sqrt[3]{x}}\right)^3 + C$

3. $\int \frac{dx}{x^2 \cdot \left(\sqrt{1+x^2}\right)^3} = \int x^{-2} (1+x^2)^{-\frac{3}{2}} dx =$

$$= \left|\begin{array}{ll} m = -2, \quad n = 2 \quad p = -\frac{3}{2} & \\ \frac{m+1}{n} + p = -\frac{1}{2} - \frac{3}{2} = -2 & \\ 3^\circ \text{ slučaj} & \\ \text{smjena: } x^{-2} + 1 = t^2 & dx = -\frac{1}{2} \cdot (t^2-1)^{-\frac{3}{2}} \cdot 2t\,dt \\ x^{-2} = t^2 - 1 & dx = -t \cdot (t^2-1)^{-\frac{3}{2}}\, dt \\ x^2 = (t^2-1)^{-1} & \\ x = (t^2-1)^{-\frac{1}{2}} & \end{array}\right|$$

$= \int (t^2-1) \cdot \left(1 + \frac{1}{t^2-1}\right)^{-\frac{3}{2}} \cdot (-t) \cdot (t^2-1)^{-\frac{3}{2}} dt =$

$= \int (t^2-1) \cdot \left(\frac{t^2-1+1}{t^2-1}\right)^{-\frac{3}{2}} \cdot (-t) \cdot (t^2-1)^{-\frac{3}{2}} dt =$

$$\int (t^2-1)\cdot\frac{t^{-3}}{(t^2-1)^{-\frac{3}{2}}}\cdot(-t)\cdot(t^2-1)^{-\frac{3}{2}}\,dt$$

$$=\int(-1+t^{-2})\,dt=-t-\frac{1}{t}+C=\frac{-t^2-1}{t}+C=$$

$$=\frac{-x^{-2}-1-1}{\sqrt{x^{-2}+1}}+C=\frac{-x^{-2}-2}{\sqrt{x^{-2}+1}}+C$$

za vježbu:

4) $\int\frac{dx}{\sqrt[3]{(1+x^2)^2}}$ (ispitni) ✓

5) $\int\frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}\,dx$ – I tip ✓

6) $\int\frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx$ – II tip ✓

7) $\int\frac{dx}{x^3\cdot\sqrt[3]{2-x^3}}$ – III tip ✓

Eulerove smjene

$I=\int R(x,\sqrt{ax^2+bx+c})\,dx$

R – racionalna funkcija

1° $\sqrt{ax^2+bx+c}=\pm\sqrt{a}\cdot x+t$ (ako je $a>0$)

2° $\sqrt{ax^2+bx+c}=xt\pm\sqrt{c}$ (ako je $c>0$)

3° $\sqrt{ax^2+bx+c}=t\cdot(x-x_1)$ (ako je $ax^2+bx+c=a\cdot(x-x_1)(x-x_2)$

$x_1,x_2\in\mathbb{R}$)

1. $I = \int \frac{dx}{x+\sqrt{x^2+x+1}}$

smjena: $\sqrt{x^2+x+1} = -x+t \quad /^2$

$$\cancel{x^2}+x+1 = \cancel{x^2}-2xt+t^2$$

$$x+2xt = t^2-1$$

$$x\cdot(1+2t) = t^2-1$$

$$x = \frac{t^2-1}{1+2t}$$

$$dx = \frac{2t\cdot(1+2t)-2\cdot(t^2-1)}{(1+2t)^2}$$

$$dx = \frac{2t+4t^2-2t^2+2}{(1+2t)^2}\,dt$$

$$dx = \frac{2t^2+2t+2}{(1+2t)^2}\,dt$$

$$I = \int \frac{\frac{2t^2+2t+2}{(1+2t)^2}}{t}\,dt =$$

$$= \int \frac{2t^2+2t+2}{t\cdot(1+2t)^2}\,dt$$

$$\frac{2t^2+2t+2}{t\cdot(1+2t)^2} = \frac{A}{t} + \frac{B}{1+2t} + \frac{C}{(1+2t)^2} \quad / \cdot t\cdot(1+2t)^2$$

$$2t^2+2t+2 = A\cdot(1+2t)^2 + Bt(1+2t) + Ct$$

$$A = 2, \quad B = -3, \quad C = -3$$

$$I = \int \frac{2}{t}\,dt - 3\int \frac{1}{1+2t}\,dt - 3\int \frac{1}{(1+2t)^2}\,dt =$$

$$= 2\ln|t| - \frac{3}{2}\ln|1+2t| + \frac{3}{2(1+2t)} + C$$

$$t = x+\sqrt{x^2+x+1}$$

$$I = \frac{dx}{1+\sqrt{1-2x-x^2}}$$

$$c' = 1$$

$$xt - 1 = \sqrt{1-2x-x^2} \quad /^2$$

$$x^2t^2 - 2xt + \cancel{1} = \cancel{1} - 2x - x^2 \quad /:x$$

$$xt^2 - 2t = -2 - x$$

$$xt^2 + x = 2t - 2$$

$$x\cdot(t^2+1) = 2t - 2$$

$$x = \frac{2t-2}{t^2+1}$$

$$dx = \frac{2(t^2+1) - 2t(2t-2)}{(t^2+1)^2}\,dt$$

$$dx = \frac{2t^2+2-4t^2+4t}{(t^2+1)^2}\,dt$$

$$dx = \frac{-2t^2+4t+2}{(t^2+1)^2}\,dt$$

$$I = \int \frac{\frac{-2t^2+4t+2}{(t^2+1)^2}}{\frac{2t-2}{t^2+1}\cdot t}\,dt = \int \frac{\cancel{2}\cdot(-t^2+2t+1)\,dt}{\cancel{2}\cdot(t-1)\cdot t\cdot(t^2+1)}$$

$$\frac{-t^2+2t+1}{t\,(t-1)(t^2+1)} = \frac{A}{t} + \frac{B}{t-1} + \frac{Ct+D}{t^2+1} \quad \dots \quad A, B, C, D \text{ ra vy.}$$

$$A = -1, \quad B = 1, \quad C = 0, \quad D = 2$$

$$= -\int \frac{1}{t}\,dt + \int \frac{1}{t-1}\,dt + \int \frac{2\,dt}{t^2+1} =$$

$$= -\ln|t| + \ln|t-1| + 2\,\mathrm{arctg}\, t + C$$

$$= \ln\left|\frac{t-1}{t}\right| + 2\,\mathrm{arctg}\, t + C$$

$$t = \frac{1+\sqrt{1-2x-x^2}}{x}$$

3. $I=\int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}\,dx$

$$x^2+3x+2=(x+1)\cdot(x+2)$$

$$\sqrt{x^2+3x+2}=t(x+1)\ /^2$$

$$x^2+3x+2=t^2(x+1)^2$$

$$(x+1)(x+2)=t^2(x+1)^2\ /:(x+1)$$

$$x+2=t^2\cdot(x+1)$$

$$x+2=t^2x+t^2$$

$$x-t^2x=t^2-2$$

$$x(1-t^2)=t^2-2$$

$$x=\frac{t^2-2}{1-t^2}$$

$$dx=\frac{2t\cdot(1-t^2)+2t\cdot(t^2-2)}{(1-t^2)^2}\,dt$$

$$dx=\frac{2t-2t^3+2t^3-4t}{(1-t^2)^2}\,dt$$

$$dx=\frac{-2t}{(1-t^2)^2}\,dt$$

$$I=\int \frac{\frac{t^2-2}{1-t^2}+\frac{t}{1-t^2}}{\frac{t^2-2}{1-t^2}-\frac{t}{1-t^2}}\cdot\frac{-2t}{(1-t^2)^2}\,dt=$$

$$\left[\sqrt{x^2+3x+2}=t\cdot\left(\frac{t^2-2}{1-t^2}+1\right)=t\cdot\frac{t^2-2+1-t^2}{1-t^2}=\frac{-t}{1-t^2}\right]$$

$$=\int\frac{(t^2+t-2)\cdot(-2t)}{(t^2-t-2)(1-t^2)^2}\,dt$$

$$=-2\int\frac{(t-1)(t+2)\,t}{(t+1)\cdot(t-2)\cdot(t-1)^2(t+1)^2}\,dt$$

$$-2\int \frac{(t^2+2t)}{(t-1)\cdot(t-2)\cdot(t+1)^3}\,dt$$

$$= \frac{-2t^2-4t}{(t-1)\cdot(t-2)\cdot(t+1)^3} = \frac{A}{t-1} + \frac{B}{t-2} + \frac{C}{t+1} + \frac{D}{(t+1)^2} + \frac{E}{(t+1)^3}$$

Za vježbu:

a) $\int \frac{1-\sqrt{x^2+x+1}}{x\cdot\sqrt{x^2+x+1}}\,dx$ (racionalisati

b) $\int x\cdot\sqrt{x^2-2x+2}\,dx$ d) $\int \frac{x^2\,dx}{(4-2x+x^2)\cdot\sqrt{2+2x-x^2}}$

c) $\int \frac{dx}{\left[1+\sqrt{x\cdot(1+x)}\right]^2}$